Independence of the Sample Mean and Variance for Normal Distributions: A Proof by Induction

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査読付き論文

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Abstract

It is well known that the sample mean and variance of a random sample drawn from a normal population independently follow normal and chi-squared distributions. The proof for the independence usually relies on the condition for the independence of two quadratic forms or a linear and a quadratic form, the orthogonal transformation, or the concept of sufficiency. This note gives a much more straightforward proof by induction without using such advanced subjects.

Keywords: normal random sample; independence of the sample mean and variance; proof by induction.
MSC 2010 Codes: 97K70 (primary); 62H10 (secondary).

1. Introduction

Let X_i, $i = 1,..., n\,(\geq 2)$ be a random sample drawn from a $N(\mu, \sigma^2)$ distribution. It is well known that the sample mean

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and variance

$$s_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2$$

have the following properties:

(a) $\bar{X}_n \sim N\left(\mu, \sigma^2/n\right)$.
(b) $W_n := (n-1)\, s_n^2/\sigma^2 \sim \chi^2_{n-1}$.
(c) $\bar{X}$ and s_n^2 are independent.

Although (a) and (b) are shown straightforwardly by the transformations of normal random variables, statistics textbooks show (c) *separately* by relying on the condition for the independence of two quadratic forms or a linear and a quadratic form (e.g. Theorem 4.17 in Graybill, 1961; Theorem 2 of Chapter 12 in Hogg and Craig, 1970; and the first proof for Theorem 3.5.1 in Tong, 1990), the orthogonal transformation (e.g. Appendix 1.5 in Hoel,1962; the second proof for Theorem 3.5.1 in Tong, 1990; and Theorem 6 of Chapter 9 in Roussas, 1997), or the concept of sufficiency (e.g. Application after Theorem 9 of Chapter 11 in Roussas, 1997). Instead, this note demonstrates (a)(b) and (c) *together* by induction. The proof is free of the aforementioned subjects, which are often beyond the scope of course work for introductory mathematical statistics. The knowledge on moment generating functions or characteristic functions is not presumed, either. It may appear that the proof requires alternative advanced subjects of differential and integral calculus including the Jacobian, the Gamma and Beta functions. However, such subjects have been applied for the proof of (b) (the derivation of the probability density function (abbreviated as “p.d.f.” hereinafter) of χ^2_{n-1}) rather than (c), and thus they are not additional prerequisites.

2. The Proof

The proof starts with the case of $n = 2$. Observe that

$$\bar{X}_2 = \frac{1}{2}\left(X_1 + X_2\right) \text{ and } W_2 = \frac{1}{\sigma^2}\left\{\left(X_1 - \bar{X}_2\right)^2 + \left(X_2 - \bar{X}_2\right)^2\right\}.$$

Solving this system for (X_1, X_2) yields

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \bar{X}_2 \pm \frac{\sigma}{\sqrt{2}}\sqrt{W_2} \\ \bar{X}_2 \mp \frac{\sigma}{\sqrt{2}}\sqrt{W_2} \end{bmatrix},$$

and thus the Jacobian for the transformation from (X_1, X_2) to $\left(\bar{X}_2, W_2\right)$ is

$$J = \det\begin{bmatrix} \partial X_1/\partial \bar{X}_2 & \partial X_1/\partial W_2 \\ \partial X_2/\partial \bar{X}_2 & \partial X_2/\partial W_2 \end{bmatrix} = \det\begin{bmatrix} 1 & \pm\sigma/\left(2\sqrt{2}\sqrt{W_2}\right) \\ 1 & \mp\sigma/\left(2\sqrt{2}\sqrt{W_2}\right) \end{bmatrix} = \mp\frac{\sigma}{\sqrt{2}}\frac{1}{\sqrt{W_2}}.$$

Then, the joint p.d.f. of $\left(\bar{X}_2, W_2\right)$ becomes

$$
\begin{aligned}
& f_{(\bar{X}_2, W_2)}(\bar{x}_2, w_2) \\
&= 2\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^2 \exp\left[-\left(\frac{1}{2\sigma^2}\right)\left\{(x_1-\mu)^2+(x_2-\mu)^2\right\}\right]\left(\frac{\sigma}{\sqrt{2}}\frac{1}{\sqrt{w_2}}\right) \\
&= 2\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^2 \exp\left[-\left(\frac{1}{2\sigma^2}\right)\left\{\sigma^2 w_2+2(\bar{x}_2-\mu)^2\right\}\right]\left(\frac{\sigma}{\sqrt{2}}\frac{1}{\sqrt{w_2}}\right) \\
&= \left[\frac{1}{\sqrt{2\pi}(\sigma/\sqrt{2})}\exp\left\{-\frac{(\bar{x}_2-\mu)^2}{2(\sigma^2/2)}\right\}\right]\left\{\frac{1}{2^{\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)}w_2^{\frac{1}{2}-1}\exp\left(-\frac{w_2}{2}\right)\right\},
\end{aligned}
$$

where $\Gamma(\alpha)=\int_0^\infty y^{\alpha-1}\exp(-y)\,dy,\ \alpha>0$ is the Gamma function. Therefore, $\bar{X}_2 \sim N(\mu, \sigma^2/2)$, $W_2=(2-1)s_2^2/\sigma^2 \sim \chi_1^2$, and $\bar{X}_2$ and W_2 (and thus s_2^2) are independent.

Next, suppose that $\bar{X}_n \sim N(\mu, \sigma^2/n)$, $W_n \sim \chi_{n-1}^2$, and $\bar{X}_n$ and W_n are independent for some $n\,(\geq 2)$. Consider the case of $n+1$. Since $(\bar{X}_n, W_n, X_{n+1})$ are mutually independent, the joint p.d.f. of $(\bar{X}_n, W_n, X_{n+1})$ is

$$
\begin{aligned}
& f_{(\bar{X}_n, W_n, X_{n+1})}(\bar{x}_n, w_n, x_{n+1}) \\
&= \left[\frac{1}{\sqrt{2\pi}(\sigma/\sqrt{n})}\exp\left\{-\frac{(\bar{x}_n-\mu)^2}{2(\sigma^2/n)}\right\}\right]\left\{\frac{1}{2^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)}w_n^{\frac{n-1}{2}-1}\exp\left(-\frac{w_n}{2}\right)\right\} \\
&\quad \times\left[\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x_{n+1}-\mu)^2}{2\sigma^2}\right\}\right]. \qquad (1)
\end{aligned}
$$

Now derive the joint p.d.f. of $(\bar{X}_{n+1}, W_{n+1})$ using (1). Let $Y_{n+1}=X_{n+1}$. Then, straightforward calculations yield the transformation from $(\bar{X}_n, W_n, X_{n+1})$ to $(\bar{X}_{n+1}, W_{n+1}, Y_{n+1})$ as

$$
\begin{bmatrix} \bar{X}_{n+1} \\ W_{n+1} \\ Y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{n+1}X_{n+1}+\frac{n}{n+1}\bar{X}_n \\ W_n+\frac{1}{(n+1)\sigma^2}\left(X_{n+1}-\bar{X}_n\right)^2 \\ X_{n+1} \end{bmatrix}.
$$

Solving this system for $(\bar{X}_n, W_n, X_{n+1})$ gives

$$
\begin{bmatrix} \bar{X}_n \\ W_n \\ X_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{n+1}{n}\bar{X}_{n+1}-\frac{1}{n}Y_{n+1} \\ W_{n+1}-\frac{n+1}{n\sigma^2}\left(Y_{n+1}-\bar{X}_{n+1}\right)^2 \\ Y_{n+1} \end{bmatrix}, \qquad (2)
$$

and thus the Jacobian for the transformation from $(\bar{X}_n, W_n, X_{n+1})$ to $(\bar{X}_{n+1}, W_{n+1}, Y_{n+1})$ is

$$
\begin{aligned}
J &= \det\begin{bmatrix} \partial\bar{X}_n/\partial\bar{X}_{n+1} & \partial\bar{X}_n/\partial W_{n+1} & \partial\bar{X}_n/\partial Y_{n+1} \\ \partial W_n/\partial\bar{X}_{n+1} & \partial W_n/\partial W_{n+1} & \partial W_n/\partial Y_{n+1} \\ \partial X_{n+1}/\partial\bar{X}_{n+1} & \partial X_{n+1}/\partial W_{n+1} & \partial X_{n+1}/\partial Y_{n+1} \end{bmatrix} \\
&= \det\begin{bmatrix} \frac{n+1}{n} & 0 & -\frac{1}{n} \\ 2\left(\frac{n+1}{n\sigma^2}\right)\left(Y_{n+1}-\bar{X}_{n+1}\right) & 1 & -2\left(\frac{n+1}{n\sigma^2}\right)\left(Y_{n+1}-\bar{X}_{n+1}\right) \\ 0 & 0 & 1 \end{bmatrix} \\
&= \frac{n+1}{n}. \qquad (3)
\end{aligned}
$$

Substituting (2) and (3) into (1) and rearranging it yield the joint p.d.f. of $\left(\bar{X}_{n+1}, W_{n+1}, Y_{n+1}\right)$ as

$$
\begin{aligned}
& f_{\left(\bar{X}_{n+1}, W_{n+1}, Y_{n+1}\right)}\left(\bar{x}_{n+1}, w_{n+1}, y_{n+1}\right) \\
= & \frac{1}{2^{\frac{n-1}{2}} \Gamma\left(\frac{n-1}{2}\right)} \frac{1}{\sqrt{2\pi}\left(\sigma/\sqrt{n}\right)} \frac{1}{\sqrt{2\pi}\sigma} \frac{n+1}{n}\left\{w_{n+1}-\frac{n+1}{n\sigma^2}\left(y_{n+1}-\bar{x}_{n+1}\right)^2\right\}^{\frac{n-1}{2}-1} \\
& \times \exp\left[-\frac{1}{2\sigma^2}\left\{\sigma^2 w_{n+1}+(n+1)\left(\bar{x}_{n+1}-\mu\right)^2\right\}\right]. \qquad (4)
\end{aligned}
$$

To integrate out Y_{n+1} from (4) to obtain the joint p.d.f. of $\left(\bar{X}_{n+1}, W_{n+1}\right)$, consider

$$
\int_{-\infty}^{\infty}\left\{w_{n+1}-\frac{n+1}{n\sigma^2}\left(y_{n+1}-\bar{x}_{n+1}\right)^2\right\}^{\frac{n-1}{2}-1} dy_{n+1}.
$$

Let $u=w_{n+1}-(n+1)\left(y_{n+1}-\bar{x}_{n+1}\right)^2/\left(n\sigma^2\right)$. Then, by

$$
y_{n+1}=\bar{x}_{n+1} \pm \sqrt{\frac{n}{n+1}}\sigma\sqrt{w_{n+1}-u} \text{ and } dy_{n+1}=\mp\sqrt{\frac{n}{n+1}}\sigma\frac{1}{2\sqrt{w_{n+1}-u}}du,
$$

the integral becomes

$$
\begin{aligned}
& \int_{-\infty}^{\infty}\left\{w_{n+1}-\frac{n+1}{n\sigma^2}\left(y_{n+1}-\bar{x}_{n+1}\right)^2\right\}^{\frac{n-1}{2}-1} dy_{n+1} \\
= & \sqrt{\frac{n}{n+1}}\sigma\int_0^{w_{n+1}} u^{\frac{n-1}{2}-1}\left(w_{n+1}-u\right)^{-\frac{1}{2}} du \\
= & \sqrt{\frac{n}{n+1}}\sigma w_{n+1}^{-\frac{1}{2}}\int_0^{w_{n+1}} u^{\frac{n-1}{2}-1}\left(1-\frac{u}{w_{n+1}}\right)^{-\frac{1}{2}} du.
\end{aligned}
$$

Furthermore, let $v=u/w_{n+1}$. Then, by $u=w_{n+1}v$, $du=w_{n+1}dv$ and the definition of the Beta function, this integral reduces to

$$
\begin{aligned}
& \sqrt{\frac{n}{n+1}}\sigma w_{n+1}^{-\frac{1}{2}}\int_0^{w_{n+1}} u^{\frac{n-1}{2}-1}\left(1-\frac{u}{w_{n+1}}\right)^{-\frac{1}{2}} du \\
= & \sqrt{\frac{n}{n+1}}\sigma w_{n+1}^{-\frac{1}{2}}\int_0^1\left(w_{n+1}v\right)^{\frac{n-1}{2}-1}(1-v)^{-\frac{1}{2}} w_{n+1}dv \\
= & \sqrt{\frac{n}{n+1}}\sigma w_{n+1}^{\frac{n}{2}-1}\int_0^1 v^{\frac{n-1}{2}-1}(1-v)^{\frac{1}{2}-1} dv \\
= & \sqrt{\frac{n}{n+1}}\sigma w_{n+1}^{\frac{n}{2}-1}\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}. \qquad (5)
\end{aligned}
$$

Then, by (4) and (5), the joint p.d.f. of $\left(\bar{X}_{n+1}, W_{n+1}\right)$ finally becomes

$$
\begin{aligned}
& f_{\left(\bar{X}_{n+1}, W_{n+1}\right)}\left(\bar{x}_{n+1}, w_{n+1}\right) \\
= & \int_{-\infty}^{\infty} f_{\left(\bar{X}_{n+1}, W_{n+1}, Y_{n+1}\right)}\left(\bar{x}_{n+1}, w_{n+1}, y_{n+1}\right) d y_{n+1} \\
= & \frac{1}{2^{\frac{n-1}{2}} \Gamma\left(\frac{n-1}{2}\right)} \frac{1}{\sqrt{2 \pi}(\sigma / \sqrt{n})} \frac{1}{\sqrt{2 \pi} \sigma} \frac{n+1}{n} \sqrt{\frac{n}{n+1}} \sigma w_{n+1}^{\frac{n}{2}-1} \frac{\Gamma\left(\frac{n-1}{2}\right) \sqrt{\pi}}{\Gamma\left(\frac{n}{2}\right)} \\
& \times \exp \left[-\frac{1}{2 \sigma^2}\left\{\sigma^2 w_{n+1}+(n+1)\left(\bar{x}_{n+1}-\mu\right)^2\right\}\right] \\
= & \left[\frac{1}{\sqrt{2 \pi}(\sigma / \sqrt{n+1})} \exp \left\{-\frac{\left(\bar{x}_{n+1}-\mu\right)^2}{2 \sigma^2 /(n+1)}\right\}\right]\left\{\frac{1}{2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} w_{n+1}^{\frac{n}{2}-1} \exp \left(-\frac{w_{n+1}}{2}\right)\right\} .
\end{aligned}
$$

Therefore, $\bar{X}_{n+1} \sim N\left(\mu, \sigma^2 /(n+1)\right)$, $W_{n+1}=\{(n+1)-1\} s_{n+1}^2 / \sigma^2 \sim \chi_n^2$, and $\bar{X}_{n+1}$ and W_{n+1} (and thus s_{n+1}^2) are independent. This completes the proof by induction.

Acknowledgments

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